Non-Abelian Duality in the String Effective Action

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Abstract

We study the symmetry of the one-loop effective action of bosonic string theory under non-Abelian T-duality transformations. It is shown that the original Lagrangian and its dual are proportional. This result implies that the corresponding reduced low energy effective actions are equivalent and leads to a functional relation between the Weyl anomaly coefficients of the original and dual two-dimensional non-linear sigma models. Finally, we apply this formalism to some simple examples.

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1   Introduction

Nowadays, duality and its different aspects are major tools in the study of string theory. One of these, target space duality or T-duality connects seemingly different string background fields [1]. It relies on the presence of isometries in the metric, the antisymmetric tensor and the dilaton field. The procedure to obtain the dual action corresponding to a given non-linear sigma model is well known [2]. It consists in gauging these isometries together with imposing, by means of a Lagrange multiplier, that the gauge field strength vanishes. The elimination of the gauge fields leads to a non-linear sigma model having the same global form as the original one but with different background fields. This means that a string propagating in a spacetime endowed with a specific background metric, for instance, presents the same dynamics when moving in another spacetime having a completely different geometry. The T-duality transformations are said to be Abelian [1] or non-Abelian [3, 4] according to the nature of the Lie algebra formed by the generators of the isometries. All of these considerations are, however, at the level of the classical theory.

One of the important questions, therefore, is whether the duality transformations remain symmetries of the quantum theory. A first step towards finding an answer to this difficult problem resides in comparing the renormalization properties of the original sigma model and its dual. One is led to considering the beta functions of the two theories. These should be studied regardless of whether the sigma model backgrounds provide consistent backgrounds in which the string can propagate (that is when the beta functions vanishes). Most of the investigations in this subject deal with Abelian T-duality. A relation between the beta functions of the original theory and their counterparts in the dual model has been given in [5]. This relation has been reached by requiring that the duality procedure commutes with the renormalization flow. In other words, one demands that $\left[T , \frac{d}{d\mu} \right] = 0$, where T is the operator implementing the duality transformations and μ is the scale of the renormalization group. At the one-loop level, the relation between the beta functions has been shown to hold for any sigma model possessing Abelian Killing vectors [5]. At two loops [6, 7, 8, 9, 10] and three loops [11], the same relation holds, provided that one modifies the duality transformations (used at the one-loop level) order by order in perturbation theory.

On the other hand, the situation regarding non-Abelian duality transformations is still far from being completely understood. It was shown in [12], using supersymmetric arguments that, if the original non-linear sigma model (with non-Abelian isometries) is conformally invariant at the one-loop level, then so is its non-Abelian dual. This proof relies on the Curci-Paffuti [13] equations which relate different Weyl anomaly coefficients (the $\bar{\beta}$-functions). Therefore not all of the relations between the $\bar{\beta}$-functions of the two theories (the original and its dual) have been found in [12]. Other explicit examples, where both the original backgrounds and their corresponding non-Abelian duals are consistent string backgrounds, have also been treated in [14, 15, 16]. It is only recently that the general relation, valid for both Abelian and non-Abelian T-duality, between the Weyl anomaly coefficients of the original theory and those corresponding to its dual has been given [17]. This relation is found through a path integral procedure and for any (not necessarily conformally invariant) non-linear sigma model. The only requirement is that the adjoint representation of the isometry Lie algebra is traceless [12, 14, 18, 19]. This relation has been shown to hold at the one-loop level for some particular examples which

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are, in general, not conformally invariant [17, 20, 21].

The aim of this paper is to check the validity of the relation between the Weyl anomaly coefficients for a general sigma model and its non-Abelian dual. Clearly, the case of the Abelian T-duality is also included here (simply by setting the structure constants of the isometry Lie algebra to zero). One of the nice properties of the Weyl anomaly coefficients of a non-linear sigma model is that they can be seen as linear combinations of the equations of motion coming from a target space action known as the string theory effective action [22]. Our strategy is to examine the behaviour of this effective action under non-Abelian T-duality. This allows us to find the precise relation between the $\bar{\beta}$-functions of the original theory and those of its non-Abelian dual. It turns out that this relation is precisely the one found in [17] through the partition function of the sigma model. This result provides another check on the intimate connection between the two-dimensional theory (where the duality transformations are found) and the target space string effective action. It is worth mentioning that the study of the string effective action under Abelian T-duality has revealed a rich structure: the duality transformations found through the sigma model are only a particular case of a family of symmetries that form an $O(d, d, R)$ group [23].

The paper is organized as follows: in the next section we recall the procedure used to obtain the dual sigma model and give the non-Abelian T-duality transformations. We show that, by choosing a suitable reparametrization of the different backgrounds, a simplified version of these transformations can be found. Furthermore, this reparametrization is essential in carrying out the explicit computation of the different terms in the original string effective action and in its dual. To get an idea of what kind of relationship one might expect between the effective action and its dual, we start, in section 3, by supposing that the functional relation of [17] between the Weyl anomaly coefficients does indeed hold. This implies that the Lagrangian of the dual string effective action must be proportional to the Lagrangian of the original action. This proportionality is then checked by a direct calculation of the dual Lagrangian, followed by a comparison with the initial Lagrangian. In section 5 we show how the formalism works for two simple examples. One of these is based on a Bianchi V type metric [24] and is known to be inequivalent to its dual at the quantum level (the adjoint representation of the isometry Lie algebra is not traceless). Nevertheless, the two reduced string effective actions (the original and its dual) are equivalent up to an overall factor. Finally, in section 7, we present our main conclusions and give possible developments of the present work. An appendix lists the essential formulas of the paper, especially those used in the explicit calculations of section 4.

2   The duality transformations

In this section our main goal is to obtain the T-duality transformations of the background fields. Let us consider a general non-linear sigma model of the form

$$
\begin{aligned}
S[x, y] \;=\; & \int d\sigma^+ d\sigma^- \{\partial_+ x^\mu Q_{\mu\nu}(x)\partial_- x^\nu + \partial_+ x^\mu Q_{\mu a}(x)e_i^a(y)\partial_- y^i \\
& + \; e_i^a(y)\partial_+ y^i Q_{a\mu}(x)\partial_- x^\mu + e_i^a(y)\partial_+ y^i Q_{ab}(x)e_j^b(y)\partial_- y^j \\
& - \; \frac{1}{4}R^{(2)}\varphi(x)\} \;.
\end{aligned}
\tag{2.1}
$$

The scalar field $\varphi(x)$ is the dilaton and $R^{(2)}$ is the two dimentional curvature of the worldsheet. As usual $\partial_\pm$ denotes differentiation with respect to $\sigma^\pm$. Here e_i^a are vielbeins defined by $(g^{-1}\partial_\pm g) = T_a e_i^a \partial_\pm y^i$ where the T_a are the generators of a Lie algebra $\mathcal{G}$ and g is an element of the corresponding Lie group. They verify

$$\partial_i e_j^a - \partial_j e_i^a = -f_{bc}^a e_i^b e_j^c, \tag{2.2}$$

where f_{bc}^a are the structure constants of $\mathcal{G}$. The inverses of these vielbeins are denoted E_a^i and satisfy $E_a^m \partial_m E_b^k - E_b^m \partial_m E_a^k = f_{ab}^c E_c^k$.

It is clear that the action (2.1) is invariant under the transformation $g \to ug$ where u is a constant element of the Lie group. As it has been done in [3, 4, 14], the dual model of the action (2.1) is found by gauging this isometry transformation. Introducing gauge fields A_+^a and A_-^a and choosing the gauge $g = 1$, yields the first order action

$$
\begin{aligned}
S_1[x, \lambda, A] \;=\; & \int d\sigma^+ d\sigma^- \{ \partial_+ x^\mu Q_{\mu\nu} \partial_- x^\nu + A_+^a Q_{a\mu} \partial_- x^\mu \\
& + \; \partial_+ x^\mu Q_{\mu a} A_-^a + A_+^a Q_{ab} A_-^b + \lambda_c (\partial_+ A_-^c - \partial_- A_+^c + f_{ab}^c A_+^a A_-^b) \\
& - \; \frac{1}{4} R^{(2)} \varphi \} \,.
\end{aligned}
\tag{2.3}
$$

The equations of motion of the Lagrange multiplier λ_c force the fields strength $F_{+-}^c = (\partial_+ A_-^c - \partial_- A_+^c + f_{ab}^c A_+^a A_-^b)$ to vanish. The solution to these constraints is $A_\pm^a = e_i^a \partial_\pm y^i$. Putting this solution into (2.3) yields the original action (2.1). On the other hand, integrating over the gauge fields, instead, leads to the dual action

$$
\begin{aligned}
\tilde{S}[x, \lambda] \;=\; & \int d\sigma^+ d\sigma^- \{ \partial_+ x^\mu Q_{\mu\nu} \partial_- x^\nu - (\partial_+ x^\mu Q_{\mu a} - \partial_+ \lambda_a) N^{ab} (Q_{b\nu} \partial_- x^\nu + \partial_- \lambda_b) \\
& - \frac{1}{4} R^{(2)} \tilde{\varphi} \} \,,
\end{aligned}
\tag{2.4}
$$

where N^{ab} is the inverse matrix of

$$M_{ab} = Q_{ab} + \lambda_c f_{ab}^c \,. \tag{2.5}$$

Notice that one can find a kind of inverse operation which permits one to recover $S_1[x, \lambda, A]$ from the dual action $\tilde{S}[x, \lambda]$. This can be seen by considering the following action

$$
\begin{aligned}
S_2[x, \lambda, C, \chi] \;=\; & \int d\sigma^+ d\sigma^- \{ \partial_+ x^\mu Q_{\mu\nu} \partial_- x^\nu - (\partial_+ x^\mu Q_{\mu a} - C_{+a}) N^{ab} (Q_{b\nu} \partial_- x^\nu + C_{-b}) \\
& - \chi_+^a (C_{-a} - \partial_- \lambda_a) + (C_{+a} - \partial_+ \lambda_a) \chi_-^a - \frac{1}{4} R^{(2)} \varphi \} \,.
\end{aligned}
\tag{2.6}
$$

The integration over the new Lagrange mutipliers $\chi_\pm^a$ results in the dual action $\tilde{S}[x, \lambda]$. However, if one keeps $\chi_\pm^a$ and integrates over C_{+a} and C_{-a}, then one arrives to the action $S_1[x, \lambda, \chi]$ where $\chi_\pm^a$ play the role of the gauge fields $A_\pm^a$. It is important to point out that this inverse operation does not rely on the existence of any isometries. This suggests that duality transformations are not necessarily related to the presence of isometries as is the case in Poisson-Lie duality [25].

In order to compare the dual action with the original one, let us make the field redefinition $\lambda_a = y^i \eta_{ia}$. For convenience, we have used the same symbol y^i to denote the original field and its dual. We choose η_{ia} to be invertible and symmetric with determinant equal

to one. To obtain the duality transformations on the backgrounds, the original sigma model is written as

$$S[x, \lambda] = \int d\sigma^+ d\sigma^- \{\partial_+ z^M (G_{MN} + B_{MN}) \partial_- z^N - \frac{1}{4} R^{(2)} \varphi\} , \tag{2.7}$$

where $z^N = (x^\mu, y^i)$. The dual sigma model is of the same form and its dual background fields are denoted $\tilde{G}_{MN}$, $\tilde{B}_{MN}$ and $\tilde{\varphi}$. The duality transformations are then given by

$$
\begin{aligned}
\tilde{G}_{\mu\nu} &= G_{\mu\nu} - \bar{S}^{ab} E_a^i E_b^j (G_{\mu i} G_{\nu j} - B_{\mu i} B_{\nu j}) + \bar{A}^{ab} E_a^i E_b^j (G_{\mu i} B_{\nu j} - B_{\mu i} G_{\nu j}) \\
\tilde{B}_{\mu\nu} &= B_{\mu\nu} - \bar{A}^{ab} E_a^i E_b^j (G_{\mu i} G_{\nu j} - B_{\mu i} B_{\nu j}) + \bar{S}^{ab} E_a^i E_b^j (G_{\mu i} B_{\nu j} - B_{\mu i} G_{\nu j}) \\
\tilde{G}_{\mu i} &= -(G_{\mu k} E_a^k \bar{A}^{ab} + B_{\mu k} E_a^k \bar{S}^{ab}) \eta_{bi} \\
\tilde{B}_{\mu i} &= -(G_{\mu k} E_a^k \bar{S}^{ab} + B_{\mu k} E_a^k \bar{A}^{ab}) \eta_{bi} \\
\tilde{G}_{ij} &= \eta_{ia} \bar{S}^{ab} \eta_{bj} \\
\tilde{B}_{ij} &= \eta_{ia} \bar{A}^{ab} \eta_{bj} \\
\tilde{\varphi} &= \varphi - \frac{1}{2} \ln \left(\det M_{ab} \right) .
\end{aligned}
\tag{2.8}
$$

Here we have written $N^{ab} = \bar{S}^{ab} + \bar{A}^{ab}$ where $\bar{S}^{ab}$ is symmetric and $\bar{A}^{ab}$ is antisymmetric. Notice that the transformation of the dilaton does not come from the classical procedure described above. In fact it is a purely quantum effect. We refer to [1, 23, 26] for details regarding the origin of this term. Independently we shall see that this dilaton shift is indeed the one needed for the equality between the reduced string effective actions of the original and the dual backgrounds.

For later use, let us write $Q_{ab} = S_{ab}(x) + v_{ab}(x)$ where S_{ab} is symmetric and v_{ab} is antisymmetric. This leads to the decomposition $M_{ab} = S_{ab} + A_{ab}$ where

$$A_{ab}(x, y) = v_{ab}(x) + y^i \eta_{ic} f_{ab}^c . \tag{2.9}$$

Using the decompositions of N^{ab} and M_{ab} together with the relation $N^{ab} M_{bc} = \delta_c^a$, we arrive at the crucial relations

$$
\begin{aligned}
S_{ab} \bar{S}^{bc} + A_{ab} \bar{A}^{bc} &= \delta_a^c \\
S_{ab} \bar{A}^{bc} + A_{ab} \bar{S}^{bc} &= 0 .
\end{aligned}
\tag{2.10}
$$

These will be used extensively in the following sections.

The T-duality transformations, as given in (2.8), are quite complicated, especially for $G_{\mu\nu}$ and $B_{\mu\nu}$. It is therefore convenient to look for a suitable decomposition of the backgrounds, in which the duality transformations are simpler. As in the Abelian case [5, 8], we choose a decomposition à la Kaluza-Klein. A reparametrization of the metric, consistent with the original sigma model, is given by

$$G_{MN} = \begin{pmatrix} g_{\mu\nu}(x) + h_{ij} V_\mu^i V_\nu^j & V_\mu^i h_{ij} \\ V_\mu^i h_{ij} & h_{ij} \end{pmatrix} \tag{2.11}$$

with

$$h_{ij}(x, y) = e_i^a(y) S_{ab}(x) e_j^b(y) , \quad V_\mu^i(x, y) = E_a^i(y) X_\mu^a(x) . \tag{2.12}$$

Similarly, the antisymmetric tensor is decomposed as

$$B_{MN} = \begin{pmatrix} b_{\mu\nu}(x) - \frac{1}{2}(V_\mu^k b_{\nu k} - V_\nu^k b_{\mu k}) & b_{\mu i} \\ -b_{\mu i} & b_{ij} \end{pmatrix} \tag{2.13}$$

where
$$b_{ij}(x,y) = e_i^a(y)v_{ab}(x)e_j^b(y) \; , \;\; b_{\mu i}(x,y) = e_i^a(y)W_{\mu a}(x) \; . \tag{2.14}$$

In this decomposition the dual backgrounds are also of the form
$$\tilde{G}_{MN} = \begin{pmatrix} \tilde{g}_{\mu\nu} + \tilde{h}_{ij}\tilde{V}_\mu^i\tilde{V}_\nu^j & \tilde{V}_\mu^i\tilde{h}_{ij} \\ \tilde{V}_\mu^i\tilde{h}_{ij} & \tilde{h}_{ij} \end{pmatrix}$$
$$\tilde{B}_{MN} = \begin{pmatrix} \tilde{b}_{\mu\nu} - \frac{1}{2}(\tilde{V}_\mu^k\tilde{b}_{\nu k} - \tilde{V}_\nu^k\tilde{b}_{\mu k}) & \tilde{b}_{\mu i} \\ -\tilde{b}_{\mu i} & \tilde{b}_{ij} \end{pmatrix} \; . \tag{2.15}$$

The transformations (2.8) become
$$\begin{aligned} \tilde{g}_{\mu\nu} &= g_{\mu\nu} \\ \tilde{b}_{\mu\nu} &= b_{\mu\nu} \\ \tilde{V}_\mu^i &= (V_\mu^l e_l^a A_{ab} - b_{\mu l}E_b^l)\eta^{bi} \\ \tilde{b}_{\mu i} &= -(V_\mu^l e_l^a S_{ab}\bar{S}^{bc} + b_{\mu l}E_b^l\bar{A}^{bc})\eta_{ci} \\ \tilde{h}_{ij} &= \eta_{ia}\bar{S}^{ab}\eta_{bj} \\ \tilde{b}_{ij} &= \eta_{ia}\bar{A}^{ab}\eta_{bj} \\ \tilde{\varphi} &= \varphi - \frac{1}{2}\ln(\det M_{ab}) \; . \end{aligned} \tag{2.16}$$

In this manner, $g_{\mu\nu}$ and $b_{\mu\nu}$ are invariants and the duality transformations act only on V_μ^i, $b_{\mu i}$, h_{ij}, b_{ij} and the dilaton φ. As it is well known, the Kaluza-Klein parametrization has other important advantages. Firstly, it enables one to calculate the inverse metric straightforwardly and, consequently, the scalar curvature. In our case, the inverse of G_{MN} is
$$G^{MN} = \begin{pmatrix} g^{\mu\nu} & -V^{\mu i} \\ -V^{\mu i} & h^{ij} + V_\mu^i V^{\mu j} \end{pmatrix} \tag{2.17}$$

where Greek indices are raised and lowered with the metric $g_{\mu\nu}$ and h^{ij} is the inverse of h_{ij}. Secondly, the determinant of the metric G_{MN}, is simply given by
$$\det G_{MN} = \det g_{\mu\nu} \det h_{ij} \; . \tag{2.18}$$

These last two properties will be useful in the next sections.

3 Weyl anomaly coefficients and string effective action

We will deal in what follows with the low energy effective action of string theory at one-loop. We will examine its behaviour under non-Abelian T-duality transformations. This effective action is given by
$$\Gamma^{(1)}[G,B,\varphi] = \int d^D z\sqrt{G}\exp(-2\varphi)\{R(G) + 4\partial_M\varphi\partial^N\varphi - \frac{1}{12}H_{MNP}H^{MNP} + \Lambda\} \; . \tag{3.1}$$

In this expression, R is the scalar curvature of the metric G_{MN} and H_{MNP}, defined by $H_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN}$, is the torsion of the antisymmetric field B_{MN}.

We have also chosen to include a cosmological constant Λ which should be set to zero for critical strings. For this task, it is more convenient to use the Kaluza-Klein decomposition of the previous section. The coordinates are split as $z^M = (x^\mu, y^i)$ where $M = 1, ..., D$, $\mu = 1, ..., d$ and $i = d+1, ..., D$.

Before getting into the details of the calculations, we will list some useful relations. Using equations (2.10) we can show, in matrix notation, that

$$M\bar{S} = SN^t \tag{3.2}$$

where N^t stands for the transpose of the matrix N^{ab}. As a direct consequence we have

$$\det M = \sqrt{\frac{\det S}{\det \bar{S}}} \ . \tag{3.3}$$

The Kaluza-Klein form of the metric G_{MN} leads to

$$\det G_{MN} = (\det e_i^c)^2 \det g_{\mu\nu} \det S_{ab} \ . \tag{3.4}$$

Similarly, the determinant of the dual metric $\tilde{G}_{MN}$ is given by

$$\det \tilde{G}_{MN} = \det g_{\mu\nu} \det \bar{S}_{ab} \tag{3.5}$$

due to the fact that we have chosen $\det \eta_{ia} = 1$. These last three equations combined with the definition of the dual dilaton result in

$$\sqrt{\det \tilde{G}} \exp(-2\tilde{\varphi}) = \frac{1}{\det e} \sqrt{\det G} \exp(-2\varphi) \ . \tag{3.6}$$

Note that the two sides of this equation are both independant of the coordinate y^i. This last expression confirms the good choice for the dilaton transformation, as it implies proportionality between the integration measures.

Furthermore, the effective action $\Gamma[G, B, \varphi] = \int d^D z \sqrt{G} \exp(-2\varphi) L$ can be expressed, up to an integration by parts, as

$$
\begin{aligned}
L[G, B, \varphi] &= R(G) - \frac{1}{12} H_{MNP} H^{MNP} - 4\partial_M\varphi \partial^M\varphi + 4\nabla_M \partial^M\varphi + \Lambda \\
&= G^{MN} \bar{\beta}_{MN}^G - 4\bar{\beta}^\varphi \ ,
\end{aligned} \tag{3.7}
$$

where the Weyl anomaly coefficients of the sigma model (2.7) are given by

$$
\begin{aligned}
\bar{\beta}_{MN}^G &= R_{MN} + 2\nabla_M \partial_N \varphi - \frac{1}{4} H_{MPQ} H_N{}^{PQ} \\
\bar{\beta}_{MN}^B &= -\frac{1}{2} \nabla^P H_{MNP} + H_{MNP} \partial^P \varphi \\
\bar{\beta}^\varphi &= -\frac{1}{2} \nabla^2 \varphi + \partial_P \varphi \partial^P \varphi - \frac{1}{24} H_{MNP} H^{MNP} - \frac{\Lambda}{4} \ .
\end{aligned} \tag{3.8}
$$

Let us turn now to a crucial relationship between the Weyl anomaly coefficients $(\bar{\beta}^G, \bar{\beta}^B, \bar{\beta}^\varphi)$ of the original sigma model and the Weyl anomaly coefficients $(\bar{\beta}^{\tilde{G}}, \bar{\beta}^{\tilde{B}}, \bar{\beta}^{\tilde{\varphi}})$ of its corresponding dual. At the one-loop level and for Abelian T-duality, these relations have been given in [5, 7]. The non-Abelian counterparts of these relations have been found in [17]. These relations have been derived using path integral arguments with

the restriction $f^a_{ab} = 0$ on the structure constants of the isometry Lie algebra [12, 18].
In order to express these relations in a compact form, we will use the generic notation
$\omega_A = (G_{MN}, B_{MN}, \varphi)$ with $\omega_1 = G_{MN}, \ldots$. Similarly the dual backgrounds are denoted
$\tilde{\omega}_A = (\tilde{G}_{MN}, \tilde{B}_{MN}, \tilde{\varphi})$. In this notation we have

$$\bar{\beta}^{\tilde{\omega}_A} = \frac{\delta\tilde{\omega}_A}{\delta\omega_B}\bar{\beta}^{\omega_B} \tag{3.9}$$

These relations have been shown to hold for a few simple sigma models. However, no
proof is known for a general sigma model. The aim of this paper is to show the validity
of equations (3.9) for a sigma model of the type considered in the previous section. For
this, we will use the string effective action.

As a starting point, let us suppose that equations (3.9) hold and let us see what the
consequences are for the effective actions $\Gamma[\omega]$ and $\tilde{\Gamma}[\tilde{\omega}]$. With the help of (3.6) we have

$$
\begin{aligned}
\tilde{G}^{MN}\bar{\beta}^{\tilde{G}}_{MN} - 4\bar{\beta}^{\tilde{\varphi}} &= \{\tilde{G}^{MN}\frac{\delta\tilde{G}_{MN}}{\delta\omega_A}\bar{\beta}^{\omega_A} - 4\frac{\delta\tilde{\varphi}}{\delta\omega_A}\} \\
&= 2\bar{\beta}^{\omega_A}\frac{\delta}{\delta\omega_A}\ln(\sqrt{\det\tilde{G}}\exp(-2\tilde{\varphi})) \\
&= \bar{\beta}^{\omega_A}\frac{\delta}{\delta\omega_A}\{\ln(\det G) - 2\varphi - \ln(\det e)\} \\
&= G^{MN}\bar{\beta}^G_{MN} - 4\bar{\beta}^{\varphi} \tag{3.10}
\end{aligned}
$$

where we have used $\frac{\delta e}{\delta\omega} = 0$. Thus we have

$$
\begin{aligned}
\Gamma[\omega] &= \int d^d x d^n y \det(e)\det(g)\det(S)\exp(-2\varphi)L[\omega] \\
\Gamma[\tilde{\omega}] &= \int d^d x d^n y \det(g)\det(S)\exp(-2\varphi)L[\omega] \tag{3.11}
\end{aligned}
$$

where $d + n = D$. Therefore, the reduced effective actions (the original and its dual)
would be proportional if one were able to perform the integration over the coordinates y^i.
This demands that $L[\omega]$ does not depend on y^i. It turns out that this is indeed the case
as explained in what follows.

From the form of the metric of the previous section it can be shown, after straightfor-
ward but tedious calculations, that

$$
\begin{aligned}
R(G) &= R(g) - \frac{1}{4}S_{ab}X^a_{\mu\nu}X^{b\mu\nu} - \frac{1}{4}S^{ab}D_\mu S_{ab}S^{cd}D^\mu S_{cd} + \frac{3}{4}S^{ab}D_\mu S_{bc}S^{cd}D^\mu S_{da} \\
&\quad - S^{ab}D_\mu D^\mu S_{ab} - \frac{1}{4}f^a_{ce}f^b_{df}S_{ab}S^{cd}S^{ef} - \frac{1}{2}f^a_{cb}f^b_{da}S^{cd} - f^a_{ca}f^b_{db}S^{cd} \tag{3.12}
\end{aligned}
$$

where

$$X^a_{\mu\nu} \equiv \partial_\mu X^a_\nu - \partial_\nu X^a_\mu + f^a_{bc}X^b_\mu X^c_\nu \tag{3.13}$$

and D_μ is the gauge covariant derivative defined by

$$D_\mu S_{ab} = \partial_\mu S_{ab} - f^c_{da}X^d_\mu S_{cb} - f^c_{db}X^d_\mu S_{ca} . \tag{3.14}$$

The expression (3.12) shows that $R(G)$ is a function of $g_{\mu\nu}(x)$, $S_{ab}(x)$ and $X^a_{\mu\nu}(x)$. Hence
it is independent of y^i.

Similarly, using the decomposition of G_{MN} in (2.11), it can be shown that

$$H^2 = H_{MNP}H^{MNP} = (h_{\mu\nu\rho}h^{\mu\nu\rho}) + 3(h_{\mu\nu i}h^{\mu\nu i}) + 3(h_{\mu ij}h^{\mu ij}) + (h_{ijk}h^{ijk}) \qquad (3.15)$$

where we raise Greek indices with $g^{\mu\nu}$ and Latin indices with $h^{ij}(x,y) = E_a^i(y)S^{ab}(x)E_b^j(y)$ with S^{ab} the inverse of S_{ab}. Here the components of h_{MNP} are defined by

$$
\begin{aligned}
h_{ijk} &= H_{ijk} \\
h_{\mu ij} &= H_{\mu ij} - V_\mu^k H_{kij} \\
h_{\mu\nu i} &= H_{\mu\nu i} - \{V_\mu^k H_{k\nu i} - (\mu \leftrightarrow \nu)\} + V_\mu^k V_\nu^l H_{kli} \\
h_{\mu\nu\rho} &= H_{\mu\nu\rho} - \{V_\mu^k H_{k\nu\rho} + \mathbf{c.p.}\} + \{V_\mu^k V_\nu^l H_{kl\rho} + \mathbf{c.p.}\} - V_\mu^k V_\nu^l V_\rho^m H_{klm} \quad (3.16)
\end{aligned}
$$

where **c.p.** stands for cyclic permutations. Using the decomposition (2.13) of the anti-symmetric tensor B_{MN}, one finds that

$$
\begin{aligned}
h_{ijk} &= e_i^a e_j^b e_k^c \{v_{ad}f_{bc}^d + \mathbf{c.p.}\} \\
h_{\mu ij} &= e_i^a e_j^b \{\partial_\mu v_{ab} + W_{\mu d}f_{ab}^d - X_\mu^c(v_{ad}f_{bc}^d + \mathbf{c.p.})\} \\
h_{\mu\nu i} &= e_i^a \{[\partial_\mu W_{\nu a} - X_\mu^b(\partial_\nu v_{ab} + W_{\nu d}f_{ab}^d) - (\mu \leftrightarrow \nu)] + X_\mu^b X_\nu^c(v_{bd}f_{ca}^d + \mathbf{c.p.})\} \\
h_{\mu\nu\rho} &= \partial_\mu(b_{\nu\rho} - \frac{1}{2}X_\nu^a W_{\rho a} + \frac{1}{2}X_\rho^a W_{\nu a}) - X_\mu^a(\partial_\nu W_{\rho a} - \partial_\rho W_{\nu a}) \\
&\quad + X_\mu^a X_\nu^b(\partial_\rho v_{ab} + W_{\rho d}f_{ab}^d) - X_\mu^a X_\nu^b X_\rho^c(v_{ad}f_{bc}^d) + \mathbf{c.p.} \qquad (3.17)
\end{aligned}
$$

We see that when performing the different contractions in (3.15), the vielbeins $e_i^a(y)$ simplify with their inverses $E_a^i(y)$ coming from h^{ij}. Hence, the dependence on y^i disappears from H^2.

Finally, the dilaton contribution is given by

$$4(\nabla_M\partial^M\varphi - \partial_M\varphi\partial^M\varphi) = 4g^{\mu\nu}\left[\nabla_\mu\partial_\nu\varphi - \partial_\mu\varphi\partial_\nu\varphi + \partial_\mu\varphi\left(\frac{1}{2}S^{ab}\partial_\nu S_{ab} + X_\nu^b f_{ba}^a\right)\right] . \quad (3.18)$$

This expression is explicitly independent of y^i. We conclude therefore that $L[\omega] = G^{MN}\bar{\beta}_{MN}^G - 4\bar{\beta}^\varphi$ is independant of y^i.

4   Invariance of the string effective action

In this section, we will prove the relation (3.9) between the Weyl anomaly coefficients of the original sigma model and those of its dual. Our strategy consists in obtaining this relation from the invariance of the string effective action under non-Abelian T-duality transformations. We begin our study from the relationship between the Weyl anomaly coefficients and the string effective action. As it is well known, these are related by

$$\bar{\beta}^{\omega_A} = M_{AB}[\omega]\frac{\delta S[\omega]}{\delta\omega_B} \qquad (4.1)$$

where the matrix M takes the form

$$M_{AB}[\omega] = -\frac{1}{\sqrt{G}\exp(-2\varphi)}\begin{pmatrix} \frac{1}{2}(G_{MP}G_{NQ} + G_{MQ}G_{NP}) & 0 & \frac{1}{4}G_{PQ} \\ 0 & \frac{1}{2}(G_{MP}G_{NQ} - G_{MQ}G_{NP}) & 0 \\ \frac{1}{4}G_{MN} & 0 & \frac{1}{16}(D-2) \end{pmatrix}$$

$$(4.2)$$

Of course the same relations hold for the dual background $\tilde{\omega}$ and we have

$$\bar{\beta}^{\tilde{\omega}_A} = M_{AB}[\tilde{\omega}] \frac{\delta S[\tilde{\omega}]}{\delta \tilde{\omega}_B} \ . \tag{4.3}$$

Suppose that we have

$$S[\tilde{\omega}] = A[\omega] S[\omega] \ \ \text{where} \ \ S[\omega] = \sqrt{G} \exp(-2\varphi) L[\omega] \ , \ \ S[\tilde{\omega}] = \sqrt{\tilde{G}} \exp(-2\tilde{\varphi}) L[\tilde{\omega}] \tag{4.4}$$

then, using a chain rule and the expressions (4.1, 4.3), we deduce that

$$\bar{\beta}^{\tilde{\omega}_A} = M_{AB}[\tilde{\omega}] \frac{\delta A[\omega]}{\delta \tilde{\omega}_B} S[\omega] + A[\omega] M_{AB}[\tilde{\omega}] \frac{\delta \omega_C}{\delta \tilde{\omega}_B} M^{CD}[\tilde{\omega}] \bar{\beta}^{\omega_D} \tag{4.5}$$

where $M^{CD}[\tilde{\omega}]$ denotes the inverse matrix of $M_{AB}[\tilde{\omega}]$. This last expression is comparable to (3.9) if one has

$$\frac{\delta A}{\delta \tilde{\omega}} = 0 \ , \ \ A[\omega] M_{AB}[\tilde{\omega}] \frac{\delta \omega_C}{\delta \tilde{\omega}_B} M^{CD}[\tilde{\omega}] = \frac{\delta \tilde{\omega}_A}{\delta \omega_D} \ . \tag{4.6}$$

Our aim is therefore to show that equations (4.4) and (4.6) are indeed satisfied. This will be our task in this section.

We start by proving that equation (4.4) is fulfilled. One cannot check this at the level of the reduced expressions as given in the previous section. This is due to the fact that in computing those reduced terms, we have thrown away some expressions whose transformed counterparts are non zero. For instance, we have, in the original effective action $\Gamma[\omega]$, terms such as $\partial_i \varphi(x)$, which obviously vanish. However, the equivalent term in the dual effective action $\Gamma[\tilde{\omega}]$ is $\partial_i \tilde{\varphi}(x,y)$, which is different from zero. This situation is to be contrasted with Abelian T-duality, where the original backgrounds and their duals are both independent of the coordinates y^i. Hence the above problem is not encountered in the Abelian case. In order to avoid this potential source of errors, one is forced to work with a string effective action where the dependence on the coordinates x^μ and y^i is not a priori specified. The particular coordinate dependence of the backgrounds is made explicit only once the duality transformations are carried out.

Checking that $S[\tilde{\omega}] = A[\omega] S[\omega]$ is a tremendous task. Here, we will give a highlight of the calculations and leave the details for the appendix. Moreover, we will write down only the expressions corresponding to the original effective action $\Gamma[\omega]$. Those corresponding to the dual effective action $\Gamma[\tilde{\omega}]$ are simply obtained by replacing each tensor by a tilded one: its T-duality transformed tensor as given in (2.16). The Kaluza-Klein decomposition of the metric G_{MN} yields the Ricci scalar[4]

$$
\begin{aligned}
R(G) \ = \ & R(g) \\
& + \{ h^{ik} h^{jl} \partial_i \partial_j h_{kl} - h^{ij} h^{kl} \partial_i \partial_j h_{kl} \\
& - \frac{1}{2} h^{ir} h^{jl} h^{ks} \partial_i h_{jk} \partial_l h_{rs} + \frac{3}{4} h^{il} h^{jr} h^{ks} \partial_i h_{jk} \partial_l h_{rs} \\
& - h^{ij} h^{kr} h^{ls} \partial_i h_{jk} \partial_l h_{rs} - \frac{1}{4} h^{il} h^{jk} h^{rs} \partial_i h_{jk} \partial_l h_{rs} + h^{ij} h^{kl} h^{rs} \partial_i h_{jk} \partial_l h_{rs} \}
\end{aligned}
$$

---

[4]We have used the package Ricci, developped by J.M. Lee, to perform most of the calculations here.

$$+g^{\mu\nu}\{(-h^{ij}\nabla_\mu\nabla_\nu h_{ij} + \frac{3}{4}h^{ik}h^{jl}\nabla_\mu h_{ij}\nabla_\nu h_{kl} - \frac{1}{4}h^{ij}h^{kl}\nabla_\mu h_{ij}\nabla_\nu h_{kl})$$

$$+(2\partial_i\nabla_\mu V^i_\nu + h^{ik}\partial_i V^j_\nu\nabla_\mu h_{jk} + 2h^{jk}V^i_\mu\nabla_\nu\partial_i h_{jk} + h^{jk}\partial_i h_{jk}\nabla_\mu V^i_\nu$$

$$+h^{jk}\partial_i V^i_\mu\nabla_\nu h_{jk} - \frac{3}{2}h^{jl}h^{kr}V^i_\mu\nabla_\nu h_{lr}\partial_i h_{jk} + \frac{1}{2}h^{jk}h^{lr}V^i_\mu\nabla_\nu h_{lr}\partial_i h_{jk})$$

$$+(-2V^j_\mu\partial_i\partial_j V^j_\nu - \frac{1}{2}\partial_i V^j_\mu\partial_j V^i_\nu - \partial_i V^i_\mu\partial_j V^j_\nu - \frac{1}{2}h^{ik}h_{jl}\partial_i V^j_\mu\partial_k V^l_\nu$$

$$-h^{il}V^k_\mu\partial_i V^j_\nu\partial_k h_{jl} - h^{kl}V^i_\mu\partial_i V^j_\nu\partial_j h_{kl} - h^{kl}V^j_\mu\partial_i V^i_\nu\partial_j h_{kl}$$

$$-h^{kl}V^i_\mu V^j_\nu\partial_i\partial_j h_{kl} - \frac{1}{2}h^{ir}h^{ks}V^j_\mu V^l_\nu\partial_i h_{jk}\partial_l h_{rs} + \frac{1}{2}h^{jl}h^{ks}V^i_\mu V^r_\nu\partial_i h_{jk}\partial_l h_{rs}$$

$$+\frac{3}{4}h^{jr}h^{ks}V^i_\mu V^l_\nu\partial_i h_{jk}\partial_l h_{rs} - \frac{1}{4}h^{jk}h^{rs}V^i_\mu V^l_\nu\partial_i h_{jk}\partial_l h_{rs})\}$$

$$-\frac{1}{4}g^{\mu\rho}g^{\nu\lambda}h_{ij}(V^i_{\mu\nu} - F^i_{\mu\nu})(V^j_{\rho\lambda} - F^j_{\rho\lambda}) \tag{4.7}$$

with the definitions

$$V^i_{\mu\nu} = \partial_\mu V^i_\nu - \partial_\nu V^i_\mu \text{ and } F^i_{\mu\nu} = V^k_\mu\partial_k V^i_\nu - V^k_\nu\partial_k V^i_\mu . \tag{4.8}$$

The terms have been assembled according to the number of factors of $g^{\mu\nu}$ and the number of factors of V^i_μ. This separation of terms will serve as a guide in our calculation. Since $g_{\mu\nu}$ is T-duality invariant then $R(g)$ is obviously invariant too.

Let us turn our attention now to the dilatonic contribution. In Abelian duality it is convenient to reparametrize the dilaton in a T-duality invariant manner [23]. This simplifies greatly the calculations. Unfortunately this happens to be impossible here due to the transformation properties of the internal metric h_{ij}. The best we can do is to reparametrize the dilaton as

$$\varphi(x) = \psi(x,y) + \theta(x,y). \tag{4.9}$$

where we define $\theta = \frac{1}{4}\ln\det h_{ij}$. Under duality transformations, we have $\varphi \to \tilde{\varphi} = \varphi - \frac{1}{4}\ln\det S + \frac{1}{4}\ln\det\bar{S}$ and $\theta \to \tilde{\theta} = \frac{1}{4}\ln\det\bar{S}$. We deduce therefore the following transformation for ψ

$$\psi \to \tilde{\psi}(x) = \psi(x,y) + \frac{1}{2}\ln\det e(y) . \tag{4.10}$$

This new scalar field ψ has an interesting property: its dual depends only on x (this can be seen by replacing, in the last expression, ψ by $\varphi - \theta$). Using the decomposition (4.9), the dialtonic contribution to the original effective action is

$$\begin{aligned}
4G^{MN}(\nabla_M\partial_N\varphi - \partial_M\varphi\partial_N\varphi) = \; &4[G^{\mu\nu}\partial_\mu\partial_\nu\psi - G^{\mu\nu}\partial_\mu\psi\partial_\nu\psi \\
&+2G^{\mu i}\partial_i\partial_\mu\psi - G^{MN}\Gamma^\lambda_{MN}\partial_\lambda\psi \\
&-2G^{\mu i}\partial_\mu\psi\partial_i\psi - 2G^{\mu M}\partial_\mu\psi\partial_M\theta \\
&+G^{ij}\partial_i\partial_j\psi + G^{MN}\partial_M\partial_N\theta \\
&-G^{MN}\Gamma^i_{MN}\partial_i\psi - G^{MN}\Gamma^P_{MN}\partial_P\theta \\
&-G^{ij}\partial_i\psi\partial_j\psi - 2G^{iM}\partial_i\psi\partial_M\theta \\
&-G^{MN}\partial_M\theta\partial_N\theta] .
\end{aligned} \tag{4.11}$$

The following expressions for the Christoffel symbols are useful

$$G^{MN}\Gamma^\lambda_{MN}(G) = g^{\mu\nu}\Gamma^\lambda_{\mu\nu}(g) + g^{\lambda\alpha}[(-\frac{1}{2}h^{ij}\nabla_\alpha h_{ij}) + (\partial_i V^i_\alpha + \frac{1}{2}V^k_\alpha h^{ij}\partial_k h_{ij})] \quad (4.12)$$

$$G^{MN}\Gamma^k_{MN}(G) = (h^{ij}h^{kl}\partial_i h_{jl} - \frac{1}{2}h^{ik}h^{jl}\partial_i h_{jl}) + g^{\alpha\beta}(\nabla_\alpha V^k_\beta + \frac{1}{2}V^k_\beta h^{ij}\nabla_\alpha h_{ij})$$

$$+ g^{\alpha\beta}[-\partial_i(V^i_\alpha V^k_\beta) - \frac{1}{2}V^i_\alpha V^k_\beta h^{jl}\partial_i h_{jl}] \quad (4.13)$$

The contribution coming from the torsion term, H^2, is determined next. The expressions in (3.16) together with the decomposition (2.13) lead to

$$h_{ijk} = \partial_i b_{jk} + \mathbf{c.p.} \quad (4.14)$$

$$h_{\mu ij} = (\partial_\mu b_{ij}) + (-V^k_\mu \partial_k b_{ij} + \{\partial_j b'_{\mu i} + (\partial_j V^k_\mu)b_{ki} - (i \leftrightarrow j)\}) \quad (4.15)$$

$$h_{\mu\nu i} = (b'_{\mu\nu i} + V^k_{\mu\nu}b_{ki}) + ([V^k_\mu b'_{i\nu k} - (\mu \leftrightarrow \nu)] - \frac{1}{2}\partial_i U_{\mu\nu} - F^k_{\mu\nu}b_{ki}) \quad (4.16)$$

$$h_{\mu\nu\rho} = (\partial_\rho b_{\mu\nu}) + (-V^k_\rho \partial_k b_{\mu\nu}) + \frac{1}{2}(V^k_{\mu\rho}b'_{\nu k} + b'_{\mu\rho k}V^k_\nu)$$

$$+ \frac{1}{2}(F^k_{\rho\mu}b'_{\nu k} + V^k_\rho V^l_\mu b'_{l\nu k}) + \mathbf{c.p.} \quad (4.17)$$

where we have defined

$$b'_{\mu i} = b_{\mu i} - V^k_\mu b_{ki}$$
$$b'_{\mu\rho i} = \partial_\mu b'_{\rho i} - \partial_\rho b'_{\mu i}$$
$$b'_{i\rho j} = \partial_i b'_{\rho j} - \partial_j b'_{\rho i}$$
$$U_{\mu\nu} = V^k_\mu b'_{\nu k} - V^k_\nu b'_{\mu k}. \quad (4.18)$$

The introduction of these new tensors is motivated by their simplified T-duality transformation rules, as seen below. In terms of these tensors we have

$$-\frac{1}{12}H_{MNP}H^{MNP} = \{+\frac{1}{2}h^{km}h^{il}h^{jn}\partial_k b_{ij}\partial_l b_{mn} - \frac{1}{4}h^{kl}h^{im}h^{jn}\partial_k b_{ij}\partial_l b_{mn}\}$$

$$+ g^{\mu\nu}h^{im}h^{jn}\{(\partial_\mu b_{ij}\partial_m b'_{\nu n} + \partial_\mu b_{ij}\partial_m V^k_\nu b_{kn}$$

$$+ \frac{1}{2}\partial_\mu b_{ij}V^k_\nu \partial_k b_{mn})$$

$$+ (-\frac{1}{4}V^k_\mu V^l_\nu \partial_k b_{ij}\partial_l b_{mn} - V^l_\mu \partial_l b_{ij}\partial_m V^k_\nu b_{kn}$$

$$- V^l_\mu \partial_l b_{mn}\partial_i b'_{\nu j} - \frac{1}{2}\partial_m V^l_\mu b_{ln}[\partial_i V^k_\nu b_{kj} - (i \leftrightarrow j)]$$

$$- \partial_m V^k_\mu b_{kn}[\partial_i b'_{\nu j} - (i \leftrightarrow j)]$$

$$- \frac{1}{2}\partial_m b'_{\mu n}[\partial_i b'_{\nu j} - (i \leftrightarrow j)])\}$$

$$- \frac{1}{4}g^{\mu\rho}g^{\nu\lambda}h^{ij}(h^{(1)}_{\mu\nu i} + h^{(2)}_{\mu\nu i})(h^{(1)}_{\rho\lambda j} + h^{(2)}_{\rho\lambda j})$$

$$- \frac{1}{12}g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}h_{\mu\nu\rho}h_{\alpha\beta\gamma} \quad (4.19)$$

where we have defined

$$h^{(1)}_{\mu\nu i} = b'_{\mu\nu i} + V^k_{\mu\nu}b_{ki} \quad , \quad h^{(2)}_{\mu\nu i} = [V^k_\mu b'_{i\nu k} - (\mu \leftrightarrow \nu)] - \frac{1}{2}\partial_i U_{\mu\nu} - F^k_{\mu\nu}b_{ki}. \quad (4.20)$$

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The terms of (4.19) have been also written according to the number of factors of $g_{\mu\nu}$, V^i_μ and $b_{\mu i}$.

Having listed the different contributions to $L[\omega]$ and $L[\tilde\omega]$, we are in a position to compare them. The easiest part to start with is the dilatonic contribution involving the field ψ in both actions. As a consequence of (4.10), we have $\partial_\mu\tilde\psi = \partial_\mu\psi$. Notice that $\partial_\mu\psi$ and $\partial_\mu\tilde\psi$ can only come from the dilatonic parts of the two actions. Therefore, by identifying these terms in the two actions using (4.11), we must have

$$G^{\mu\nu}\partial_\mu\partial_\nu\psi - G^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + 2G^{\mu i}\partial_i\partial_\mu\psi - G^{MN}\Gamma^\lambda_{MN}\partial_\lambda\psi - 2G^{\mu i}\partial_\mu\psi\partial_i\psi - 2G^{\mu M}\partial_\mu\psi\partial_M\theta =$$
$$\tilde G^{\mu\nu}\partial_\mu\partial_\nu\tilde\psi - \tilde G^{\mu\nu}\partial_\mu\tilde\psi\partial_\nu\tilde\psi + 2\tilde G^{\mu i}\partial_i\partial_\mu\tilde\psi - \tilde G^{MN}\tilde\Gamma^\lambda_{MN}\partial_\lambda\tilde\psi - 2\tilde G^{\mu i}\partial_\mu\tilde\psi\partial_i\tilde\psi - 2\tilde G^{\mu M}\partial_\mu\tilde\psi\partial_M\tilde\theta \quad (4.21)$$

The first two terms on each side of this equality are equal since $G^{\mu\nu} = \tilde G^{\mu\nu} = g^{\mu\nu}$. The third term on each side vanishes due to the particular dependence on x^μ and y^i of ψ and $\tilde\psi$. An explicit calculation of the remaining terms leads to

$$\frac{1}{2}V^k_\sigma h^{ij}\partial_k h_{ij} + \partial_k V^k_\sigma = \partial_k\tilde V^k_\sigma \ . \qquad (4.22)$$

Using the explicit form of $\tilde V^k_\sigma$, as given in (2.16), we find that this last equation is satisfied only when

$$V^k_\sigma e^b_k f^a_{ab} = 0 \qquad (4.23)$$

This equation holds if the isometry Lie algebra is such that $f^a_{ab} = 0$, as is the case for semi-simple Lie algebras. This observation agrees with the conclusions reached in [12, 14, 18, 19] for the quantum validity of non-Abelian T-duality transformations. However, the terms containing f^a_{ab} contain always V^k_σ at the same time. Therefore, for a block-diagonal metric with $V^k_\sigma = 0$ the above equation is also satisfied. This particular case will be discussed in an example based on a Bianchi-type metric in the next section.

The other particular terms to consider in comparing $\Gamma[\omega]$ and $\Gamma[\tilde\omega]$ are the contributions involving three factors of $g^{\mu\nu}$. These are found in the expressions of H^2 and $\tilde H^2$. According to (4.19), we must have

$$g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}h_{\mu\nu\rho}h_{\alpha\beta\gamma} = \tilde g^{\mu\alpha}\tilde g^{\nu\beta}\tilde g^{\rho\gamma}\tilde h_{\mu\nu\rho}\tilde h_{\alpha\beta\gamma} \ . \qquad (4.24)$$

Since $g^{\mu\nu} = \tilde g^{\mu\nu}$, we must have

$$h_{\mu\nu\rho} = \tilde h_{\mu\nu\rho} \ . \qquad (4.25)$$

We have checked, using the list of transformations given in the appendix, that this is indeed the case.

The comparison of the remaining terms in $L[\omega]$ and $L[\tilde\omega]$ is organised according to the following pattern: If we look at the transformations (2.16), we note two important properties. First, since the reduced metric $g^{\mu\nu}$ is invariant, the terms in the two Lagrangians with the same number of factors of $g^{\mu\nu}$ must be equal. Moreover, $\tilde V^i_\mu$ and $\tilde b_{\mu i}$ contain one V^i_μ and one $b_{\mu i}$ factors. Therefore, the terms in $L[\tilde\omega]$, consisting of n factors of $\tilde V^i_\mu$ and m factors of $\tilde b_{\mu i}$ can only be compared with terms in $L[\omega]$ containing n' factors of $\tilde V^i_\mu$ and m' factors of $\tilde b_{\mu i}$, such that $n' + m' = n + m$. Using this criterion as a guiding principle, we have explicitly checked that $L[\tilde\omega] = L[\omega]$. We have listed in an appendix the transformations of all the terms.

Having shown that $S[\tilde\omega] = A[\omega]S[\omega]$, with $A[\omega] = 1/\det e$, we can now turn our attention to proving the relations in (4.6). It is clear that $\frac{\delta A[\omega]}{\delta\tilde\omega_B} = \frac{\delta A[\omega]}{\delta\omega_A}\frac{\delta\omega_A}{\delta\tilde\omega_B} = 0$. In order

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to deal with the second equation in (4.6), let us start by showing that all the matrices involved there are well defined. The right hand side requires the computation of $\frac{\delta \tilde{\omega}_A}{\delta \omega_B}$. For this, one needs to compute the variations δS_{ab} and δA_{ab}. Using the key relations (2.10), we find

$$
\begin{aligned}
\frac{\delta \bar{S}^{ab}}{\delta S_{cd}} &= -\frac{1}{2}[\bar{S}^{ac}\bar{S}^{db} + \bar{A}^{ac}\bar{A}^{db} + (c \leftrightarrow d)] \\
\frac{\delta \bar{S}^{ab}}{\delta A_{cd}} &= -\frac{1}{2}[\bar{S}^{ac}\bar{A}^{db} + \bar{A}^{ac}\bar{S}^{db} - (c \leftrightarrow d)] \\
\frac{\delta \bar{A}^{ab}}{\delta S_{cd}} &= -\frac{1}{2}[\bar{S}^{ac}\bar{A}^{db} + \bar{A}^{ac}\bar{S}^{db} + (c \leftrightarrow d)] \\
\frac{\delta \bar{A}^{ab}}{\delta A_{cd}} &= -\frac{1}{2}[\bar{S}^{ac}\bar{S}^{db} + \bar{A}^{ac}\bar{A}^{db} - (c \leftrightarrow d)] \ .
\end{aligned}
\tag{4.26}
$$

These expressions, together with $\frac{\delta e_i^a}{\delta \omega_A} = 0$, and the duality transformations (2.8) yield the different components of $\frac{\delta \tilde{\omega}_A}{\delta \omega_B}$. These are set forth in the appendix.

The matrix $M^{CD}[\omega]$, inverse of $M_{AB}[\omega]$, is computed from the variation of the action, where $\frac{\delta S[\omega]}{\delta \omega_C} = M^{CD}[\omega]\bar{\beta}^{\omega_D}$ and is given by

$$
M^{CD}[\omega] = -\sqrt{G} \exp(-2\varphi) \times
\begin{pmatrix}
\frac{1}{2}(G^{MP}G^{NQ} + G^{MQ}G^{NP} - G^{MN}G^{PQ}) & 0 & 2G^{PQ} \\
0 & \frac{1}{2}(G^{MP}G^{NQ} - G^{MQ}G^{NP}) & 0 \\
2G^{MN} & 0 & 8
\end{pmatrix}
$$

On the other hand, the computation of $\frac{\delta \omega_C}{\delta \tilde{\omega}_B}$ requires the inversion of the duality transformations. The original string backgrounds can expressed in terms of the dual ones as follows

$$
\begin{aligned}
G_{\mu\nu} &= \tilde{G}_{\mu\nu} - S_{ab}\eta^{ai}\eta^{bj}(\tilde{G}_{\mu i}\tilde{G}_{\nu j} - \tilde{B}_{\mu i}\tilde{B}_{\nu j}) + A_{ab}\eta^{ai}\eta^{bj}(\tilde{G}_{\mu i}\tilde{B}_{\nu j} - \tilde{B}_{\mu i}\tilde{G}_{\nu j}) \\
B_{\mu\nu} &= \tilde{B}_{\mu\nu} - A_{ab}\eta^{ai}\eta^{bj}(\tilde{G}_{\mu i}\tilde{G}_{\nu j} - \tilde{B}_{\mu i}\tilde{B}_{\nu j}) + S_{ab}\eta^{ai}\eta^{bj}(\tilde{G}_{\mu i}\tilde{B}_{\nu j} - \tilde{B}_{\mu i}\tilde{G}_{\nu j}) \\
G_{\mu i} &= -(\tilde{G}_{\mu k}\eta^{ka}A_{ab} + \tilde{B}_{\mu k}\eta^{ka}S^{ab})e_i^b \\
B_{\mu i} &= -(\tilde{G}_{\mu k}\eta^{ka}S_{ab} + \tilde{B}_{\mu k}\eta^{ka}A^{ab})e_i^b \\
G_{ij} &= e_i^a S_{ab} e_j^b \\
B_{ij} &= e_i^a A_{ab} e_j^b - y^k \eta_{kc} f_{ab}^c e_i^a e_j^b \\
\varphi &= \tilde{\varphi} - \frac{1}{2} \ln \det(\bar{S}^{ab} + \bar{A}^{ab}) \ .
\end{aligned}
\tag{4.27}
$$

The different functionnal derivatives are then calculated in a similar way to those of $\frac{\delta \tilde{\omega}_A}{\delta \omega_B}$ (recall that $\frac{\delta}{\delta \tilde{\omega}_B}(y^k \eta_{kc} f_{ab}^c e_i^a e_j^b) = 0$).

Since the inverses of the matrices $M_{AB}[\omega]$ and $\frac{\delta \omega_A}{\delta \tilde{\omega}_B}$ exist, the second relation in (4.6) can be put in the form

$$
A[\omega]M_{AB}[\tilde{\omega}] = \frac{\delta \tilde{\omega}_A}{\delta \omega_C} M_{CD}[\omega]\frac{\delta \tilde{\omega}_D}{\delta \omega_B}
\tag{4.28}
$$

Using the explicit form of $M_{AB}[\omega]$ in (4.28) and the expressions of $\frac{\delta \tilde{\omega}_A}{\delta \omega_B}$ given in the appendix, we have checked that the above equation is indeed satisfied. Let us just give an example of how this works. The $(\tilde{G}_{ij}, \tilde{\varphi})$ component of the left hand side equation (4.28) is

$$
A[\omega]M_{\tilde{G}_{ij},\tilde{\varphi}} = -\frac{1}{4 \det e \sqrt{\tilde{G}} \exp(-2\tilde{\varphi})}\tilde{G}_{ij}
\tag{4.29}
$$

while the left hand side is written as

$$\frac{\delta \tilde{G}_{ij}}{\delta G_{mn}} M_{G_{mn},\varphi} \frac{\delta \tilde{\varphi}}{\delta \varphi} + \frac{\delta \tilde{G}_{ij}}{\delta G_{mn}} M_{G_{mn},G_{pq}} \frac{\delta \tilde{\varphi}}{\delta G_{pq}} + \frac{\delta \tilde{G}_{ij}}{\delta B_{mn}} M_{B_{mn},B_{pq}} \frac{\delta \tilde{\varphi}}{\delta B_{pq}} \tag{4.30}$$

where

$$
\begin{aligned}
M_{G_{mn},\varphi} &= -\frac{1}{4\sqrt{G}\exp(-2\varphi)} G_{mn} \\
M_{G_{mn},G_{pq}} &= -\frac{1}{2\sqrt{G}\exp(-2\varphi)} (G_{mp}G_{nq} + G_{mq}G_{np}) \\
M_{B_{mn},B_{pq}} &= -\frac{1}{2\sqrt{G}\exp(-2\varphi)} (G_{mp}G_{nq} - G_{mq}G_{np}) \, .
\end{aligned}
\tag{4.31}
$$

With the help of the functionnal derivatives of the appendix, one verifies that the two sides are equal. This is also the case for all the other components of (4.28).

5   Explicit examples

In this section we study two examples regarding the relationship between the original effective action and its dual. The first is a model based on the isometry algebra $SU(2)$. We consider the sigma model given by following action [17]

$$S = \int d^2\sigma \left\{ f(x)\partial_\mu x \partial^\mu x + h(x) Tr\left[\left(g^{-1}\partial_\mu g\right)\left(g^{-1}\partial^\mu g\right)\right]\right\} \tag{5.1}$$

where g is an element of the Lie group $SU(2)$ and $f(x)$ and $h(x)$ are arbitrary functions. We take the elements of $SU(2)$ to be parametrized by the Euler angles $y^i = (u, v, w)$

$$g = \exp(iu\tau_3)\exp(iv\tau_1)\exp(iw\tau_3) \tag{5.2}$$

where $\tau_i = \sigma_i/2$ with σ_i the Pauli matrices. The only non-vanishing background field is the metric and it is given by

$$G_{MN} = \begin{pmatrix} f(x) & 0 \\ 0 & h_{ij}(x,y) \end{pmatrix} \tag{5.3}$$

with

$$h_{ij}(x,y) = h(x) \begin{pmatrix} 1 & 0 & \cos(v) \\ 0 & 1 & 0 \\ \cos(v) & 0 & 1 \end{pmatrix} . \tag{5.4}$$

The vielbeins are given by

$$e_i^a(y) = \begin{pmatrix} \sin(v)\sin(w) & \cos(w) & 0 \\ -\sin(v)\cos(w) & \sin(w) & 0 \\ \cos(v) & 0 & 1 \end{pmatrix} \tag{5.5}$$

where the index a numbers the rows of this matrix.

The objects of the previous sections needed here are $S_{ab}(x) = h(x)\delta_{ab}$ and $\eta_{ia} f^a_{bc} = \varepsilon_{ibc}$ with $\varepsilon_{123} = 1$. These allow the determination of $\bar{S}^{ab}$ and $\bar{A}^{ab}$. The effective action corresponding to this sigma model is found to be given by

$$\begin{aligned}
\Gamma[\omega] &= \int dx d^3 y \sqrt{G} R(G) \\
&= \frac{3}{2} \int dx\, du\, dv\, dw\, |\sin(v)|\, f^{-\frac{3}{2}} h^{\frac{1}{2}} \left(f^2 - 2f \frac{d^2 h}{dx^2} + \frac{df}{dx}\frac{dh}{dx} \right) .
\end{aligned} \tag{5.6}$$

Notice that one can perform the integration over the internal coordinates and obtain the internal volume $v = \int du\, dv\, dw\, |\sin(v)|$. The dual background fields are computed from the duality transformations (2.16). This yields the following metric, antisymmetric tensor and dilaton

$$\tilde{G}_{MN} = \frac{1}{hD} \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & h^2 + u^2 & uv & uw \\ 0 & vu & h^2 + v^2 & vw \\ 0 & wu & wv & h^2 + w^2 \end{pmatrix}$$

$$\tilde{B}_{MN} = \frac{1}{D} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -w & v \\ 0 & w & 0 & -u \\ 0 & -v & u & 0 \end{pmatrix}$$

$$\tilde{\varphi} = -\frac{1}{2}\ln(\det M_{ab}) = -\frac{1}{2}\ln(hD) . \tag{5.7}$$

where $D = h^2 + u^2 + v^2 + w^2$. Using these expressions one can see by direct calculations that the T-dual effective action is given by

$$\begin{aligned}
\Gamma[\tilde{\omega}] &= \int dx d^3 y \sqrt{\tilde{G}} \exp(-2\tilde{\varphi}) \left\{ \tilde{R} - \frac{1}{12}\tilde{H}^2 + 4\left(\tilde{\nabla}\tilde{\partial}\tilde{\varphi} - (\tilde{\partial}\tilde{\varphi})^2 \right) \right\} \\
&= \int dx\, du\, dv\, dw\, f^{-\frac{3}{2}} h^{\frac{1}{2}} \left(f^2 - 2f \frac{d^2 h}{dx^2} + \frac{df}{dx}\frac{dh}{dx} \right) .
\end{aligned} \tag{5.8}$$

Similarly, the volume element of the internal space in the dual theory is $\tilde{v} = \int du\, dv\, dw$. Therefore, the two reduced effective actions $\Gamma[\omega]$ and $\Gamma[\tilde{\omega}]$ are equal up to an overall factor.

Our second example is based on a Bianchi V type metric. This example was first analysed in [24] where the authors found that the $\bar{\beta}$-functions of the original model and its dual are not equivalent. The original model is characterized by

$$G_{MN} = \mathrm{diag}(-f(x), a^2(x), a^2(x)e^{-2u}, a^2(x)e^{-2u}) , \quad B_{MN} = \varphi = 0 \tag{5.9}$$

where the coordinates are such that $z^M = (x, y^i) = (x, u, v, w)$. The vielbeins of the Bianchi V metric are

$$e^a_i = \mathrm{diag}(1, e^{-u}, e^{-u}) \tag{5.10}$$

and the non-vanishing structure constants corresponding to the isometry group are

$$f^2_{12} = -f^2_{21} = f^3_{13} = -f^3_{31} = 1 . \tag{5.11}$$

This example is particularly interesting as f_{ab}^a is different from zero, while V_μ^i is zero. Therefore equation (4.23) is satisfied and we expect the equality of the reduced effective actions. We will see that this is the case here. However some complications arise at the level of the equations of motion (the $\bar{\beta}$-function equations). By a similar method to the one used in the $SU(2)$ example, the dual backgrounds are given by

$$
\tilde{G}_{MN} = \begin{pmatrix} -f & 0 & 0 & 0 \\ 0 & a^4/D & 0 & 0 \\ 0 & 0 & (a^4 + w^2)/D & -vw/D \\ 0 & 0 & -vw/D & (a^4 + v^2)/D \end{pmatrix}
$$

$$
\tilde{B}_{MN} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a^2v/D & -a^2w/D \\ 0 & a^2v/D & 0 & 0 \\ 0 & a^2w/D & 0 & 0 \end{pmatrix}
$$

$$
\tilde{\varphi} = -\frac{1}{2}\ln(D) \tag{5.12}
$$

where $D = a^2(x)\,(a^4(x) + v^2 + w^2)$. With these original and dual backgrounds, the corresponding effective actions are

$$
\Gamma[\omega] = 3\int dx\,du\,dv\,dw\,e^{-2u}f^{-\frac{1}{2}}a\left(f\frac{d^2a^2}{dx^2} - a\frac{da}{dx}\frac{df}{dx} - 2f^2\right)
$$

$$
\Gamma[\tilde{\omega}] = 3\int dx\,du\,dv\,dw\,f^{-\frac{1}{2}}a\left(f\frac{d^2a^2}{dx^2} - a\frac{da}{dx}\frac{df}{dx} - 2f^2\right) \ . \tag{5.13}
$$

We have then proportionality between the reduced effective actions. However, this does not mean that the $\bar{\beta}$-functions are related according to (3.9). Notice that although $L[\omega] = L[\tilde{\omega}]$, one does not, in general, have $L[\omega + \delta\omega] = L[\tilde{\omega} + \delta\tilde{\omega}]$. The difference between $L[\omega + \delta\omega]$ and $L[\tilde{\omega} + \delta\tilde{\omega}]$ always involves terms containing $f_{ab}^a e_i^a \delta V_\mu^i$. Clearly, these terms vanish only when $f_{ab}^a = 0$. This is why the beta functions of the original and dual theories cannot be related by (3.9) in this case. This explains the conclusions reached in [24] for the particular case $f(x) = 1$ and $a(x) = x$.

6 Discussion

We have analysed in this paper the effects of non-Abelian T-duality transformations on the Weyl anomaly coefficients of the two-dimensional non-linear sigma model. This analysis is carried out at the level of the corresponding string effective action. A suitable reparametrization of the string backgrounds, in which the duality transformations take a simple form, is found. In this reparametrization the dual backgrounds are of the same global form as the original ones. By an explicit computation, we show that the Lagrangian of the original string effective action is proportional to that of the dual string effective action. Using then the connection between the $\bar{\beta}$-functions of the non-linear sigma model and the equations of motion of the string effective action, we give a functional relation between the Weyl anomaly coefficients of the original two-dimensional theory and their counterparts in the dual one. This is precisely the relation found in [17] through path integral considerations.

A natural question arises now regarding the invertibility of the non-Abelian duality transformations. At the level of the non-linear sigma model, it is well known that these are not invertible, as the dual model does not possess any isometries which allow one to go back to the original theory [14]. However, when dealing with the string effective action, one can consider the transformations (2.8) in their own right, regardless of their two-dimensional origin. This is indeed what happens in the Abelian case [23]. Let us therefore examine this important question. In order to recover the diagonal part of the metric of the original theory, namely h_{ij}, from the dual one, that is $\tilde{h}_{ij}$, we must impose the following duality transformations

$$
\begin{aligned}
\bar{S}^{cd} &\rightarrow \eta^{ci} e_i^a S_{ab} e_j^b \eta^{jd} \\
S_{cd} &\rightarrow \eta_{ci} E_a^i \bar{S}^{ab} E_b^j \eta_{jd} \\
e_i^a &\rightarrow \eta_{ib} e_k^b \eta^{ka} \\
E_a^i &\rightarrow \eta^{ib} E_b^k \eta_{ka}
\end{aligned}
\tag{6.1}
$$

This results in the chain of duality transformations $h_{ij} \rightarrow \tilde{h}_{ij} = \eta_{ia} \bar{S}^{ab} \eta_{bj} \rightarrow \tilde{h}_{ij} = e_i^a S_{ab} e_j^b$. However, these transformations for S_{ab} and $\bar{S}_{ab}$ must preserve the relations (2.10). This requires that A_{ab} and $\bar{A}_{ab}$ must transform as

$$
\begin{aligned}
\bar{A}^{cd} &\rightarrow \eta^{ci} e_i^a A_{ab} e_j^b \eta^{jd} \\
A_{cd} &\rightarrow \eta_{ci} E_a^i \bar{A}^{ab} E_b^j \eta_{jd}
\end{aligned}
\tag{6.2}
$$

Using the above duality transformation we automatically get

$$
\begin{aligned}
G_{\mu i} &\rightarrow -(G_{\mu k} E_a^k \bar{A}^{ab} + B_{\mu k} E_a^k \bar{S}^{ab}) \eta_{bi} \rightarrow G_{\mu i} \\
B_{\mu i} &\rightarrow -(G_{\mu k} E_a^k \bar{S}^{ab} + B_{\mu k} E_a^k \bar{A}^{ab}) \eta_{bi} \rightarrow B_{\mu i} \\
G_{\mu\nu} &\rightarrow \tilde{G}_{\mu\nu} \rightarrow G_{\mu\nu} \\
B_{\mu\nu} &\rightarrow \tilde{B}_{\mu\nu} \rightarrow B_{\mu\nu}
\end{aligned}
\tag{6.3}
$$

So far, all this seems to work quite well but two problems arise: Firstly, the only requirement left for the duality transformation to be invertible is

$$
b_{ij} = e_i^a v_{ab} e_j^b \rightarrow \eta_{ia} \bar{A}^{ab} \eta_{bj} \rightarrow b_{ij}
\tag{6.4}
$$

Combining this requirement with the above transformation of $A_{ab} = v_{ab} + y^k \eta_{kc} f_{ab}^c$, implies the strange transformation relation

$$
y^k \eta_{kc} f_{ab}^c e_i^a e_j^b \rightarrow 0 \rightarrow y^k \eta_{kc} f_{ab}^c e_i^a e_j^b
\tag{6.5}
$$

Secondly, using the transformations in (6.1), we deduce the following chain of transformations for the dilaton

$$
\varphi \rightarrow \varphi + \frac{1}{4} \left(\det \bar{S} - \det S \right) \rightarrow \varphi + \ln \det e
\tag{6.6}
$$

Therefore, one cannot recover the original dilaton. It is only in the Abelian case that these two problems do not arise ($f_{bc}^a = 0$ and $\det e = 1$) and the duality transformations are indeed invertible.

Our results could be of interest in cosmological models based on the string effective action. This is due to the fact that most of the relevant models in cosmology, exhibit non-Abelian rather than Abelian isometries. Thus, non-Abelian T-duality transformations could be especially useful in the study of inflationary models derived from string theories. This is in the same spirit as pre-big-bang inflation [27], where Abelian duality has proven to be an essential tool.

Furthermore, our decomposition of the different string backgounds has allowed for an explicit computation of the reduced string effective action. It is therefore natural to ask if this reduced action presents other symmetries like the $O\left(\mathrm{d},\mathrm{d}\right)$ transformations of the Abelian case. This is a crucial point which deserves further study. It is worth mentioning that a decomposition, sharing some of the features presented here, has been used in the context of massive supergravity theories [28]. The authors of that article show that the reduced low enegy effective action of the heterotic string possesses an $O\left(\mathrm{d},\mathrm{d}+16\right)$ symmetry group. However, these transformations present two drawbacks. First, the transformations involve the field strenght of the gauge fields coming from the Kaluza-Klein decomposition of the metric. This means that these symmetries are non-local on the basic fields. Secondly, the symmetry holds only when a transformation is associated to the structure constants of the isometry Lie algebra.

There are obviously other subjects to be treated in this area. As our analyses are valid only at the one-loop level, it is therefore desirable to examine what happens beyond this leading order. The one-loop duality transformations must certainly be modified. This could give an idea on how to correct the duality symmetry, loop by loop in perturbation theory. Finally, one could extend the investigation to the case of Poisson-Lie duality. This last point is currently under investigation.

A   Useful Identities

In section 4 of this paper many quantities are redefined for the sake of their simple duality transformations. In this appendix, we give the dual expressions corresponding to these quantities. Before doing this, however, one needs other useful identities used in the computation.

By applying a differential operator ∂ on both sides of the fundamental relations (2.10) one finds

$$
\begin{aligned}
\partial \bar{S}^{ab} &= -\partial S_{cd}(S^{ac}S^{db} + A^{ac}A^{db}) - \partial A_{cd}(S^{ac}A^{db} + A^{ac}S^{db}) \\
\partial \bar{A}^{ab} &= -\partial S_{cd}(S^{ac}A^{db} + A^{ac}S^{db}) - \partial A_{cd}(S^{ac}S^{db} + A^{ac}A^{db}) \ .
\end{aligned} \tag{A.1}
$$

When ∂ acts on the coordinates x^μ then we have $\partial_\mu A_{cd} = \partial_\mu v_{cd}$ while when acting on the coordinates y^i we get $\partial_i S_{cd} = 0$ and $\partial_i A_{cd} = \eta_{ia} f^a_{cd}$. To identify terms involving h^{ij} and h_{ij} in the dual effective action, we use

$$
\begin{aligned}
e^a_i(S_{ac}\bar{S}^{cd}S_{db} - A_{ac}\bar{S}^{cd}A_{db})e^b_j &= e^a_i S_{ab} e^b_j = h_{ij} \\
E^i_a(\bar{S}^{ab} - \bar{A}^{ac}S_{cd}\bar{A}^{db})E^j_b &= E^i_a S^{ab} E^j_b = h^{ij}.
\end{aligned} \tag{A.2}
$$

Furthermore, the combinations $V^k_\mu e^a_k$, $b_{\mu k}E^k_a$ and $b'_{\mu k}E^k_a$ are independant of y^i. Hence, we have the following identities

$$
\begin{aligned}
\partial_i V^j_\mu &= V^k_\mu e^a_k \partial_i E^j_a \\
\partial_i b_{\mu j} &= b_{\mu k} E^k_a \partial_i e^a_j \\
\partial_i b'_{\mu j} &= b'_{\mu k} E^k_a \partial_i e^a_j \ .
\end{aligned} \tag{A.3}
$$

Notice that the Jacobi identity together with the definition of A_{ab} implies

$$
A_{cd}f^d_{ab} + A_{ad}f^d_{bc} + A_{bd}f^d_{ca} = v_{cd}f^d_{ab} + \mathbf{c.p.} \tag{A.4}
$$

The above identities lead to the following duality transformations

$$
\begin{aligned}
\tilde{V}^i_\mu &= (V^l_\mu e^a_l y^k \eta_{kc} f^c_{ab} - b'_{\mu l} E^l_b)\eta^{bi} \\
\tilde{b}'_{\mu i} &= -V^m_\mu e^a_m \eta_{ai} \\
\tilde{b}'_{\mu\nu i} &= -V^m_{\mu\nu} e^a_m \eta_{ai} \\
\tilde{b}'_{i\mu j} &= 0 \\
\tilde{U}_{\mu\nu} &= -U_{\mu\nu} - 2F^k_{\mu\nu} e^a_k \lambda_a \\
\tilde{V}^i_{\mu\rho} &= (V^k_{\mu\rho} e^a_k A_{ac} e^c_m - h^{(1)}_{\mu\rho m}) E^m_b \eta^{bi} \\
\tilde{h}^{(1)}_{\mu\nu i} &= -(V^k_{\mu\nu} e^a_k S_{ac} \bar{S}^{cb} + h^{(1)}_{\mu\nu k} E^k_a \bar{A}^{ab})\eta_{bi} \\
\tilde{F}^i_{\mu\nu} &= (F^k_{\mu\nu} e^a_k A_{ab} e^b_m + h^{(2)}_{\mu\nu m}) E^m_b \eta^{bi} \\
\tilde{h}^{(2)}_{\mu\nu i} &= (F^k_{\mu\nu} e^a_k S_{ac} \bar{S}^{cb} - h^{(2)}_{\mu\nu k} E^k_a \bar{A}^{ab})\eta_{bi} \ .
\end{aligned} \tag{A.5}
$$

B   Comparing the original and dual Lagrangians

As mentionned in the main body of the paper, the terms in $L[\omega]$ and $L[\tilde{\omega}]$ are organised according to the number of factors of $g^{\mu\nu}$ and the number of factors of the gauge fields V^i_μ, $b_{\mu i}$. We will gather below these terms according to this criterion. The expressions coming from the dual Lagrangian $L[\tilde{\omega}]$ are given on the right hand side of each equation.

B.1 Order zero in $g^{\mu\nu}$ and zero in gauge fields

To this order one has to collect from the two Lagrangians the terms without gauge fields. Many of the terms coming from the Ricci scalar cancel against terms from the dilatonic part (note that this is the case for other orders as well). The dual expressions of the remainig terms at this order are

$$4\tilde{h}^{ij}\partial_i\partial_j\tilde{\psi} - 4\tilde{h}^{ij}\partial_i\tilde{\psi}\partial_j\tilde{\psi} + 4\partial_i(\tilde{h}^{ij}\partial_j\tilde{\psi}) = 0 \ ,$$

$$-\partial_i\partial_j\tilde{h}^{ij} = -S^{ab}f^c_{ad}f^d_{bc} - S^{ab}f^c_{ca}f^d_{db} \ ,$$

$$\frac{1}{2}\tilde{h}^{km}\tilde{h}^{il}\tilde{h}^{jn}(\partial_k\tilde{h}_{ij}\partial_l\tilde{h}_{mn} - \partial_k\tilde{b}_{ij}\partial_l\tilde{b}_{mn}) = \frac{1}{2}(S^{ab}f^c_{ad}f^d_{bc} + S^{ab}A_{cg}S^{gf}A_{eh}S^{hd}f^c_{ad}f^e_{bf}) \ ,$$

$$-\frac{1}{4}\tilde{h}^{kl}\tilde{h}^{im}\tilde{h}^{jn}(\partial_k\tilde{h}_{ij}\partial_l\tilde{h}_{mn} + \partial_k\tilde{b}_{ij}\partial_l\tilde{b}_{mn}) = \frac{1}{4}(S_{ab} - A_{ac}S^{cd}A_{db})(S^{ef}f^a_{fg}S^{gh}f^b_{he})$$

Similarly, the corresponding expressions at the same order coming from the original Lagrangian can be easily extracted from the expressions given in the main text. The sums of these terms in each Lagrangian are identical. This will be the case at each order as well.

B.2 Order one in $g^{\mu\nu}$ and zero in gauge fields

$$-\frac{1}{4}\tilde{g}^{\mu\nu}\tilde{h}^{im}\tilde{h}^{jn}(\partial_\mu\tilde{h}_{ij}\partial_\nu\tilde{h}_{mn} + \partial_\mu\tilde{b}_{ij}\partial_\nu\tilde{b}_{mn}) = -\frac{1}{4}g^{\mu\nu}S^{ac}S^{bd}(\partial_\mu S_{ab}\partial_\nu S_{cd} + \partial_\mu v_{ab}\partial_\nu v_{cd})$$

$$= -\frac{1}{4}g^{\mu\nu}h^{im}h^{jn}(\partial_\mu h_{ij}\partial_\nu h_{mn} + \partial_\mu b_{ij}\partial_\nu b_{nm})$$

B.3 Order one in $g^{\mu\nu}$ and one in gauge fields

To this order one must demand that

$$\partial_k V^{\alpha k} - 2V^{\alpha k}\partial_k\psi = 0 \tag{B.1}$$

in order to get rid of the terms involving the field ψ. Note that this is equivalent to imposing equation (4.23). The duals of the remaining terms at this order are

$$\tilde{g}^{\alpha\beta}\partial_i\tilde{V}^j_\beta\partial_\alpha\tilde{h}_{jk}\tilde{h}^{ki} + \tilde{g}^{\alpha\beta}\tilde{h}^{im}\tilde{h}^{jn}\partial_\alpha\tilde{h}_{ij}\partial_m\tilde{V}^k_\beta\tilde{b}_{kn}$$
$$= g^{\alpha\beta}V^i_\alpha e^a_i f^b_{ac}(S^{cd}A_{be}S^{ef}\partial_\beta A_{fd} - S^{cd}\partial_\beta S_{db}) \ ,$$

$$\frac{1}{2}\tilde{V}^{\alpha k}\tilde{h}^{im}\tilde{h}^{jn}\partial_\alpha\tilde{h}_{ij}\partial_k\tilde{h}_{mn} + \frac{1}{2}\tilde{g}^{\alpha\beta}\tilde{h}^{im}\tilde{h}^{jn}\partial_\alpha\tilde{h}_{ij}\tilde{V}^k_\beta\partial_k\tilde{b}_{mn}$$
$$= \frac{1}{2}g^{\mu\nu}\tilde{V}^i_\mu S^{ab}S^{cd}\partial_\nu A_{ac}\partial_i A_{bd} \ ,$$

$$\tilde{g}^{\alpha\beta}\tilde{h}^{im}\tilde{h}^{jn}\partial_\alpha\tilde{h}_{ij}\partial_m\tilde{b}'_{\beta n} = 0 \tag{B.2}$$

B.4 Order one in $g^{\mu\nu}$ and two in gauge fields

$$-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_i\tilde{V}^j_\mu\partial_j\tilde{V}^i_\nu = -\frac{1}{2}g^{\mu\nu}V^k_\mu e^a_k V^l_\nu e^b_l f^c_{ad}f^d_{bc}$$

$$-\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{h}^{ik}\tilde{h}_{jl}\partial_i\tilde{V}^j_\mu\partial_k\tilde{V}^l_\nu \quad - \quad \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{h}^{im}\tilde{h}^{jn}\partial_i\tilde{V}^l_\mu\tilde{b}_{lj}(\partial_m\tilde{V}^k_\nu\tilde{b}_{kn}-(m\leftrightarrow n))$$

$$= -\frac{1}{2}g^{\mu\nu}V^k_\mu e^a_k V^l_\nu e^b_l f^c_{ae}f^d_{bf}(S_{cd}-A_{cg}S^{gh}A_{hd})S^{ef}$$

$$+\frac{1}{2}g^{\mu\nu}V^k_\mu e^a_k V^l_\nu e^b_l f^c_{ae}f^d_{bf}A_{cg}S^{gf}A_{dh}S^{he}\ ,$$

$$-\tilde{g}^{\mu\nu}\tilde{h}^{il}\tilde{V}^k_\mu\partial_k\tilde{h}_{lj}\partial_i\tilde{V}^j_\nu \quad - \quad \tilde{g}^{\mu\nu}\tilde{h}^{im}\tilde{h}^{jn}\tilde{V}^l_\mu\partial_l\tilde{b}_{ij}\partial_m\tilde{V}^k_\nu\tilde{b}_{kn}$$

$$= g^{\mu\nu}S^{cd}S^{ef}(b'_{\mu k}E^k_a V^l_\nu e^b_l f^a_{ce}f^g_{bf}A_{gd}$$

$$+V^k_\mu e^a_k V^l_\nu e^b_l y^i\eta_{ij}f^j_{ag}f^g_{de}A_{ch}f^h_{bf})\ ,$$

$$-\frac{1}{4}\tilde{V}^i_\mu\tilde{V}^j_\nu\tilde{h}^{mp}h^{nq}\partial_i\tilde{h}_{mn}\partial_j\tilde{h}_{pq} \quad - \quad \frac{1}{4}\tilde{g}^{\mu\nu}\tilde{h}^{im}\tilde{h}^{jn}\tilde{V}^k_\mu\tilde{V}^l_\nu\partial_k\tilde{b}_{ij}\partial_l\tilde{b}_{mn}$$

$$= -\frac{1}{4}g^{\mu\nu}S^{ac}S^{bd}b'_{\mu i}E^i_e f^e_{ab}b'_{\nu j}E^j_f f^f_{cd}$$

$$-g^{\mu\nu}S^{cd}S^{ef}b'_{\mu k}E^k_a V^l_\nu e^b_l f^a_{ce}f^g_{bf}y^i\eta_{ij}f^j_{gd}$$

$$-\frac{1}{4}g^{\mu\nu}S^{cd}S^{ef}V^k_\mu e^a_k V^l_\nu e^b_l y^p\eta_{pq}f^q_{ag}f^g_{ce}y^i\eta_{ij}f^j_{bh}f^h_{df}\ ,$$

$$\tilde{g}^{\mu\nu}\tilde{h}^{im}\tilde{h}^{jn}\tilde{V}^l_\mu\partial_l\tilde{b}_{mn}\partial_i\tilde{b}'_{\nu j} \quad = \quad 0$$

$$\tilde{g}^{\mu\nu}\tilde{h}^{im}\tilde{h}^{jn}\partial_m\tilde{V}^k_\nu\tilde{b}_{kn}(\partial_i\tilde{b}'_{\nu j}-(i\leftrightarrow j)) \quad = \quad 0$$

$$-\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{h}^{im}\tilde{h}^{jn}\partial_i\tilde{b}'_{\mu j}(\partial_m\tilde{b}'_{\nu n}-(m\leftrightarrow n)) \quad = \quad 0 \tag{B.3}$$

B.5 Order two in $g^{\mu\nu}$

Here we can treat at the same time order two, three and four in the gauge fields. Using the transformations (A.5) and the relations (A.2) we can easily show that

$$\tilde{h}_{ij}(\tilde{V}^i_{\mu\nu}-\tilde{F}^i_{\mu\nu})(\tilde{V}^j_{\lambda\sigma}-\tilde{F}^j_{\lambda\sigma}) \quad + \quad \tilde{h}^{ij}(\tilde{h}^{(1)}_{\mu\nu i}+\tilde{h}^{(2)}_{\mu\nu i})(\tilde{h}^{(1)}_{\lambda\sigma j}+\tilde{h}^{(2)}_{\lambda\sigma j})$$

$$= (V^i_{\mu\nu}-F^i_{\mu\nu})e^a_i(S_{ac}\bar{S}^{cd}S_{db}-A_{ac}\bar{S}^{cd}A_{db})e^b_j(V^j_{\lambda\sigma}-F^j_{\lambda\sigma})$$

$$+(h^{(1)}_{\mu\nu i}+h^{(2)}_{\mu\nu i})E^i_a(\bar{S}^{ab}-\bar{A}^{ac}S_{cd}\bar{A}^{db})E^j_b(h^{(1)}_{\lambda\sigma j}+h^{(2)}_{\lambda\sigma j})$$

$$= h_{ij}(V^i_{\mu\nu}-F^i_{\mu\nu})(V^j_{\lambda\sigma}-F^j_{\lambda\sigma})+h^{ij}(h^{(1)}_{\mu\nu i}+h^{(2)}_{\mu\nu i})(h^{(1)}_{\lambda\sigma j}+h^{(2)}_{\lambda\sigma j})$$

B.6 Order three in $g^{\mu\nu}$: the invariance of $h_{\mu\nu\rho}$

Starting from the definition of $h_{\mu\nu\rho}$ given in (3.16) and using the decomposition (2.13) one can show that

$$h_{\mu\nu\rho}=\partial_\rho b_{\mu\nu}-V^k_\rho\partial_k b_{\mu\nu}+\frac{1}{2}(V^k_{\mu\rho}b'_{\nu k}+b'_{\mu\rho k}V^k_\nu)+\frac{1}{2}(F^k_{\rho\mu}b'_{\nu l}+V^k_\rho V^l_\mu b'_{l\nu k})+\mathbf{c.p.} \tag{B.4}$$

Now using the transformations (A.5) and the key relations (2.10) we find that

$$\frac{1}{2}(\tilde{V}^k_{\mu\rho}\tilde{b}'_{\nu k}+\tilde{b}'_{\mu\rho k}\tilde{V}^k_\nu)+\mathbf{c.p.} \quad = \quad \frac{1}{2}(V^k_{\mu\rho}b'_{\nu k}+b'_{\mu\rho k}V^k_\nu)+\mathbf{c.p.}$$

$$\frac{1}{2}(\tilde{F}^k_{\rho\mu}\tilde{b}'_{\nu k}+\tilde{V}^k_\rho\tilde{V}^l_\mu\tilde{b}'_{l\nu k})+\mathbf{c.p.} \quad = \quad \frac{1}{2}(F^k_{\rho\mu}b'_{\nu k}+V^k_\rho V^l_\mu b'_{l\nu k})+\mathbf{c.p.}$$

Moreover, the term $b_{\mu\nu}$ is invariant and independant of y^i. All this applied to the decomposition (B.4) proves the invariance of $h_{\mu\nu\rho}$ under non-Abelian T-duality.

C   Functional derivatives

Here, we list the non-vanishing components of $\frac{\delta\tilde{\omega}_A}{\delta\omega_B}$. These are needed to prove equation (4.28). With the two definitions

$$
\begin{aligned}
Y^n_\mu &= (G_{\mu i}E^i_a\bar{S}^{ab} + B_{\mu i}E^i_a\bar{A}^{ab})E^n_b \\
Z^n_\mu &= (B_{\mu i}E^i_a\bar{S}^{ab} + G_{\mu i}E^i_a\bar{A}^{ab})E^n_b
\end{aligned}
\tag{C.1}
$$

we have

$$
\frac{\delta\tilde{G}_{\mu\nu}}{\delta G_{\lambda\sigma}} = \frac{1}{2}[\delta^\lambda_\mu\delta^\sigma_\nu + (\mu \leftrightarrow \nu)]
$$

$$
\frac{\delta\tilde{G}_{\mu\nu}}{\delta G_{\lambda n}} = -[Y^n_\mu\delta^\lambda_\nu + (\mu \leftrightarrow \nu)]
$$

$$
\frac{\delta\tilde{G}_{\mu\nu}}{\delta B_{\lambda n}} = [Z^n_\mu\delta^\lambda_\nu + (\mu \leftrightarrow \nu)]
$$

$$
\frac{\delta\tilde{G}_{\mu\nu}}{\delta G_{mn}} = \frac{1}{2}\{Y^m_\mu Y^n_\nu - Z^m_\mu Z^n_\nu + (\mu \leftrightarrow \nu)\}
$$

$$
\frac{\delta\tilde{G}_{\mu\nu}}{\delta B_{mn}} = \frac{1}{2}\{Z^m_\mu Y^n_\nu - Y^m_\mu Z^n_\nu - (\mu \leftrightarrow \nu)\}
$$

$$
\frac{\delta\tilde{G}_{\mu i}}{\delta G_{\lambda n}} = -\delta^\lambda_\mu E^n_a\bar{A}^{ab}\eta_{bi}
$$

$$
= \frac{\delta\bar{B}_{\mu i}}{\delta B_{\lambda n}}
$$

$$
\frac{\delta\tilde{G}_{\mu i}}{\delta B_{\lambda n}} = -\delta^\lambda_\mu E^n_a\bar{S}^{ab}\eta_{bi}
$$

$$
= \frac{\delta\bar{B}_{\mu i}}{\delta G_{\lambda n}}
$$

$$
\frac{\delta\tilde{G}_{\mu i}}{\delta G_{mn}} = \frac{1}{2}\{Z^m_\mu E^n_c\bar{S}^{cd}\eta_{di} + Y^m_\mu E^n_c\bar{A}^{cd}\eta_{di} + (m \leftrightarrow n)\}
$$

$$
\frac{\delta\tilde{G}_{\mu i}}{\delta B_{mn}} = \frac{1}{2}\{Y^m_\mu E^n_c\bar{S}^{cd}\eta_{di} + Z^m_\mu E^n_c\bar{A}^{cd}\eta_{di} - (m \leftrightarrow n)\}
$$

$$
\frac{\delta\tilde{G}_{ij}}{\delta G_{mn}} = -\frac{1}{2}\{\eta_{ia}[\bar{S}^{ab}E^m_b E^n_c\bar{S}^{cd} + \bar{A}^{ab}E^m_b E^n_c\bar{A}^{cd}]\eta_{dj} + (m \leftrightarrow n)\}
$$

$$
\frac{\delta\tilde{G}_{ij}}{\delta B_{mn}} = -\frac{1}{2}\{\eta_{ia}[\bar{S}^{ab}E^m_b E^n_c\bar{A}^{cd} + \bar{A}^{ab}E^m_b E^n_c\bar{S}^{cd}]\eta_{dj} - (m \leftrightarrow n)\}
$$

$$
\frac{\delta\tilde{\varphi}}{\delta B_{mn}} = -\frac{1}{2}E^m_a\bar{S}^{ab}E^n_b
$$

$$
\frac{\delta\tilde{B}_{\mu\nu}}{\delta B_{\lambda\sigma}} = \frac{1}{2}[\delta^\lambda_\mu\delta^\sigma_\nu - (\mu \leftrightarrow \nu)]
$$

$$
\frac{\delta\tilde{B}_{\mu\nu}}{\delta G_{\lambda n}} = -[Z^n_\mu\delta^\lambda_\nu - (\mu \leftrightarrow \nu)]
$$

$$
\frac{\delta\tilde{B}_{\mu\nu}}{\delta B_{\lambda n}} = [Y^n_\mu\delta^\lambda_\nu - (\mu \leftrightarrow \nu)]
$$

$$\frac{\delta \tilde{B}_{\mu\nu}}{\delta B_{mn}} = \frac{1}{2}\{Y_\mu^m Y_\nu^n - Z_\mu^m Z_\nu^n - (\mu \leftrightarrow \nu)\}$$

$$\frac{\delta \tilde{B}_{\mu\nu}}{\delta G_{mn}} = \frac{1}{2}\{Z_\mu^m Y_\nu^n - Y_\mu^m Z_\nu^n - (\mu \leftrightarrow \nu)\}$$

$$\frac{\delta \tilde{B}_{\mu i}}{\delta B_{mn}} = \frac{1}{2}\{Z_\mu^m E_c^n \bar{S}^{cd}\eta_{di} + Y_\mu^m E_c^n \bar{A}^{cd}\eta_{di} - (m \leftrightarrow n)\}$$

$$\frac{\delta \tilde{B}_{\mu i}}{\delta G_{mn}} = \frac{1}{2}\{Y_\mu^m E_c^n \bar{S}^{cd}\eta_{di} + Z_\mu^m E_c^n \bar{A}^{cd}\eta_{di} + (m \leftrightarrow n)\}$$

$$\frac{\delta \tilde{B}_{ij}}{\delta B_{mn}} = -\frac{1}{2}\{\eta_{ia}[\bar{S}^{ab} E_b^m E_c^n \bar{S}^{cd} + \bar{A}^{ab} E_b^m E_c^n \bar{A}^{cd}]\eta_{dj} - (m \leftrightarrow n)\}$$

$$\frac{\delta \tilde{B}_{ij}}{\delta G_{mn}} = -\frac{1}{2}\{\eta_{ia}[\bar{S}^{ab} E_b^m E_c^n \bar{A}^{cd} + \bar{A}^{ab} E_b^m E_c^n \bar{S}^{cd}]\eta_{dj} + (m \leftrightarrow n)\}$$

$$\frac{\delta \tilde{\varphi}}{\delta B_{mn}} = \frac{1}{2}E_a^m \bar{A}^{ab} E_b^n$$

$$\frac{\delta \tilde{\varphi}}{\delta \varphi} = 1 . \tag{C.2}$$

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